Equal Binomial Coefficients: Some Elementary Considerations

Benjamin M. M. de Weger*

*Department of Mathematics, University of Leiden; and Econometric Institute,
Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands*

1. INTRODUCTION

In the Pascal Triangle, consisting of the binomial coefficients $\binom{n}{k}$ for $n = 0, 1, 2, \ldots$ and $0 \leqslant k \leqslant n$, one encounters each natural number (with the exception of 2) at least twice, and many numbers more than twice. There are three well-known relations that account for this, namely

$$\binom{n}{k} = \binom{n}{n-k}, \qquad \binom{n}{0} = 1, \qquad \binom{n}{1} = n$$

for $n = 1, 2, \ldots, 0 \leqslant k \leqslant n$. Notice that the third relation above implies

$$\binom{\binom{n}{k}}{1} = \binom{n}{k},$$

so that there are infinitely many numbers occurring at least 4 times in the Pascal Triangle.

Stripped of these trivialities, the more interesting problem becomes to determine the natural numbers that occur at least twice as binomial coefficients of the shape $\binom{n}{k}$ with $2 \leqslant k \leqslant \frac{1}{2}n$, and this is yet unsolved in its full generality. The only nontrivial solutions known at this time are the following:

$$\binom{16}{2} = \binom{10}{3} = 120, \qquad\qquad \binom{21}{2} = \binom{10}{4} = 210,$$

$$\binom{56}{2} = \binom{22}{3} = 1540, \qquad\qquad \binom{120}{2} = \binom{36}{3} = 7140,$$

$$\binom{153}{2} = \binom{19}{5} = 11628, \qquad\qquad \binom{221}{2} = \binom{17}{8} = 24310,$$

$$\binom{78}{2} = \binom{15}{5} = \binom{14}{6} = 3003,$$

and an infinite family:

$$\binom{F_{2i+2}F_{2i+3}}{F_{2i}F_{2i+3}} = \binom{F_{2i+2}F_{2i+3}-1}{F_{2i}F_{2i+3}+1} \qquad \text{for} \quad i = 1, 2, ...,$$

where F_n is the nth Fibonacci number (defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n = 1, 2, ...$). This infinite family is due to D. A. Lind [L] and D. Singmaster [Sin2].

There are no other nontrivial solutions of $\binom{n}{k} = \binom{m}{l}$ with $\binom{n}{k} \leqslant 10^{30}$ or $\max\{n, m\} \leqslant 1000$, as we could show without difficulties in a few hours on a personal computer. Notice that D. Singmaster [Sin2] searched up to $2^{48} \approx 2.8 \times 10^{14}$. We did this computer search as follows. To start with, all solutions to $\binom{n}{k} = \binom{m}{l}$ with $\max\{k, l\} \leqslant 4$, are known, see below. Next, we made a list of all $\binom{n}{k} \leqslant 10^{30}$ with $5 \leqslant k \leqslant \frac{1}{2}n$, and sorted this list. Thus numbers occurring twice in the list are easily found. Next, for each member of the list we checked whether they were of the form $\binom{m}{l}$ for $l = 2, 3, 4$ (which was the most time-consuming step). All these computations were done in exact (i.e. 30 digit) arithmetic. Finally, we made a list of all $\binom{n}{k} > 10^{30}$ with $\max\{n, m\} \leqslant 1000$ in 8 digit precision only, sorted this list, and checked for pairs being close enough.

Let $N(a)$ be the number of occurrences of a as a binomial coefficient. Then $N(1) = \infty$, $N(2) = 1$, and clearly $2 \leqslant N(a) < \infty$ for all $a \geqslant 3$. D. Singmaster [Sin1] proved that $N(a) = O(\log a)$, and conjectured that $N(a) = O(1)$. Later [Sin2] he even conjectured that $N(a) \leqslant 10$ for all $a \geqslant 2$. H. L. Abbott, P. Erdös and D. Hanson [AEH] showed that the average and normal order of $N(a)$ is 2, and that $N(a) = O(\log a / \log \log a)$. Maybe even the following is (too good to be) true.

Conjecture A. The equation $\binom{n}{k} = \binom{m}{l}$ has no nontrivial solutions but those given above.

This conjecture would imply $N(a) \leqslant 8$ for all $a \geqslant 2$, and $N(a) \leqslant 6$ for all $a \geqslant 2$ with the exception of $a = 3003$, where the upper bound $N(a) = 6$ is attained infinitely often.

In this note we will contribute a little bit to the knowledge on this conjecture, and show that the special case $\binom{n}{3} = \binom{m}{4}$ has essentially been settled over 30 years ago by L. J. Mordell, without anybody having realized this (so it seems). The special cases $\binom{n}{2} = \binom{m}{3}$ and $\binom{n}{2} = \binom{m}{4}$ have been settled before, but by much more complicated methods than we (and Mordell) need. Further, we will also prove a partial result on *rational* solutions of $\binom{n}{3} = \binom{m}{4}$. We restrict ourselves entirely to elementary methods; i.e., the deepest mathematics we require are only the first essentials of algebraic number theory.

2. INTEGRAL SOLUTIONS

In the context of diophantine equations, it's a bit more natural to study the equation $\binom{n}{k} = \binom{m}{l}$ for the extended definition of $\binom{n}{k}$ to all $n, k \in \mathbb{Z}$ with $k \geqslant 0$, as follows:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

In this more general sense (and, by the way, also in the restricted sense), for fixed k, l (with $k < l$) the equation $\binom{n}{k} = \binom{m}{l}$ has been completely solved in two cases only, namely the case $(k, l) = (2, 3)$ by È. T. Avanesov [Av] using Skolem's method, and the case $(k, l) = (2, 4)$, after Richard K. Guy had drawn attention to the problem in Section D3 of [G], by the present author [dW] and independently by Ákos Pintér [Pi], both using the Gelfond–Baker method.

It is the first purpose of this note to show that the case $(k, l) = (3, 4)$ is comparatively easy, as it is an almost trivial consequence of the result of L. J. Mordell [M1], which itself has a more or less elementary proof. In this paper, Mordell determines the products of 2 consecutive integers that are equal to products of 3 consecutive integers. It is quite remarkable that this connection between Mordell's well known result and our binomial diophantine equation seems to have been unnoticed for over 30 years.

So here's our first main theorem, which might come as a disappointment to the reader expecting nontrivialities.

THEOREM 1. *The only solutions $n, m \in \mathbb{Z}$ to $\binom{n}{3} = \binom{m}{4}$ are the following trivial ones*:

$$(n, m) \in \{0, 1, 2\} \times \{0, 1, 2, 3\}, \qquad (n, m) = (3, 4), (3, -1), (7, 7), (7, -4).$$

Proof. Write out the equation $\binom{n}{3} = \binom{m}{4}$ as

$$\tfrac{1}{6}n(n-1)(n-2) = \tfrac{1}{24}m(m-1)(m-2)(m-3).$$

Suggested by symmetry we put

$$X = n-1, \qquad Y = \tfrac{1}{2}m(m-3),$$

and then we obtain the equation

$$Y^2 + Y = X^3 - X.$$

In other words, we are looking for products $Y(Y+1)$ of two consecutive integers being equal to products $(X-1)X(X+1)$ of three consecutive integers. Mordell's Theorem 2 below gives us all the solutions for (X, Y), which are easily traced back to the trivial solutions for (n, m) given above. This completes the proof. ∎

THEOREM 2 (Mordell, 1963).   *The only solutions in* $X, Y \in \mathbb{Z}$ *to the equation*

$$Y^2 + Y = X^3 - X$$

are the following 10:

$$(X, Y) \in \{-1, 0, 1\} \times \{-1, 0\}, \quad (X, Y) = (2, 2), (2, -3), (6, 14), (6, -15).$$

Theorem 2 was proved in an elementary way by L. J. Mordell [M1] in 1963 (see also Theorem 2 in Chapter 27 of his book [M2]). By "elementary" we mean that the deepest results that are used are the explicit knowledge of a class group and generators of a unit group in a certain cubic number field. For a different approach, that seems to be more complicated, see Exercise 9.13 of J. H. Silverman's book [Sil]. We mention that a third line of proof (using much more machinery, both theoretical and computational) is made possible by the recent method of elliptic logarithms, developed independently by R. J. Stroeker and N. Tzanakis [ST], by J. Gebel, A. Pethö and H. G. Zimmer [GPZ], and by N. P. Smart [Sm]. Below we will return to Mordell's proof.

3. RATIONAL SOLUTIONS

Note that we can even extend the definition of $\binom{n}{k}$ further, to $n \in \mathbb{Q}$ (of course we can just as well take $n \in \mathbb{R}$, or even $n \in \mathbb{C}$, but we do not want to leave the area of number theory). When we want to study the equation

$\binom{n}{k} = \binom{m}{l}$ for fixed k, l in this context, we enter the domain of arithmetic algebraic geometry.

In the case $(k, l) = (2, 3)$ the equation $\binom{n}{2} = \binom{m}{3}$ is a Weierstraß equation of an elliptic curve. This curve has trivial torsion, and rank 2, and the group of rational points is generated by $(n, m) = (1, 0)$ and $(n, m) = (1, 1)$. Now, using the addition law on the elliptic curve, one can start producing the infinitely many rational solutions. In other words, the set of solutions $n, m \in \mathbb{Q}$ of $\binom{n}{2} = \binom{m}{3}$ is infinite, but well understood.

In the case $(k, l) = (2, 4)$ the equation $\binom{n}{2} = \binom{m}{4}$ also is an equation of an elliptic curve. This curve has a torsion group of order 2, generated by $(n, m) = (0, 1)$, and the free part of the group of rational points is of rank 2, and is generated by $(n, m) = (0, 0)$ and $(n, m) = (1, 1)$. Thus again the set of solutions $n, m \in \mathbb{Q}$ of $\binom{n}{2} = \binom{m}{4}$ is infinite, but well understood.

In the case $(k, l) = (3, 4)$ things are different, because the algebraic curve defined by the equation $\binom{n}{3} = \binom{m}{4}$ has genus 3, and thus, by Faltings's work [F], has only finitely many rational points. It is notoriously difficult to solve such problems of explicit determination of rational or integral points on curves of genus > 1. That we succeeded above in proving our Theorem 1 on the integral points on our curve $\binom{n}{3} = \binom{m}{4}$, is due to the remarkable fact that this curve is (in geometric language) a double cover of an elliptic curve, namely the one given by Mordell's equation $Y^2 + Y = X^3 - X$ (this is just a reformulation of our proof of Theorem 1 above). The rational points on this elliptic curve are again not too difficult to describe, in fact, that's what Silverman uses in his Exercise 9.13 referred to above. The curve has trivial torsion, rank 1, and the group of rational points is generated by $(X, Y) = (0, 0)$.

It is an interesting challenge to find out, e.g. on the basis of the facts mentioned above, what can be said about the set of rational points on the curve $\binom{n}{3} = \binom{m}{4}$. With Apecs we searched for solutions coming from the rational points $N \cdot (0, 0)$ on the elliptic curve $Y^2 + Y = X^3 - X$, for $|N| \leqslant 50$ only (but note that the numerator and denominator of the second coordinate of $50 \cdot (0, 0)$ are already numbers of about 85 digits). So we feel safe to formulate the following guess.

Conjecture B. The only solutions $n, m \in \mathbb{Q}$ to $\binom{n}{3} = \binom{m}{4}$, besides the integral ones given in Theorem 1 above, are

$$(n, m) = (\tfrac{5}{4}, \tfrac{1}{2}), (\tfrac{5}{4}, \tfrac{5}{2}).$$

It is the second theme of this note to extend Mordell's elementary proof of Theorem 2 [M1] to make a first step towards the solution of this problem. Our extension concerns so-called S-integral solutions, i.e. rational

solutions of which the denominators have prime divisors from a fixed finite set of primes only. We now restrict ourselves to the set consisting of the prime 2. Thus we have the following result.

THEOREM 3.   *The only solutions $n, m \in \mathbb{Q}$ of which the denominators are powers of 2 to the equation $\binom{n}{3} = \binom{m}{4}$, are the ones given in Conjecture B above.*

Note that this result extends Theorem 1. Following the above proof of Theorem 1, it is clear that Theorem 3 is a consequence of the following result, which is an analogous extension to the *S*-integral case of Mordell's Theorem 2.

THEOREM 4.   *The only solutions $X, Y \in \mathbb{Q}$ of which the denominators are powers of 2 to the equation $Y^2 + Y = X^3 - X$, besides the integral ones given in Theorem 2 above, are*

$$(X, Y) = \left(\frac{1}{4}, \frac{-5}{8}\right), \left(\frac{1}{4}, \frac{-3}{8}\right), \left(\frac{161}{16}, \frac{-2065}{64}\right), \left(\frac{161}{16}, \frac{2001}{64}\right).$$

4. PROOF OF THEOREM 4

We will now prove Theorem 4, partly following, and partly extending the line of argument in Mordell's original proof of Theorem 2 [M1]. Note that our proof is completely elementary.

Proof of Theorem 4.   We see at once that there is an integer $k \geqslant 0$, and integers X_1, Y_1 such that

$$X = \frac{X_1}{2^{2k}}, \qquad Y = \frac{Y_1}{2^{3k}}.$$

Then the equation $Y^2 + Y = X^3 - X$ leads to

$$Y_1^2 + 2^{3k} Y_1 = X_1^3 - 2^{4k} X_1.$$

The idea is to complete the square in the left hand side of the equation, and then factor both sides in the ring of integers of an appropriate number field. For convenience we put

$$U = 2X_1, \qquad V = 2Y_1 + 2^{3k},$$

and so obtain the equation

$$2V^2 = U^3 - 2^{4k+2}U + 2^{6k+1}, \tag{1}$$

in which the left hand side has the obvious factorization $2 \times V \times V$ over $\mathbb{Z}$.

Let θ be any root of the polynomial $u^3 - 4u + 2$. Then the right hand side of equation (1) factors over the ring of integers $\mathcal{O}_\mathbb{K}$ of the cubic number field $\mathbb{K} = \mathbb{Q}(\theta)$ as

$$U^3 - 2^{4k+2}U + 2^{6k+1} = (U - \theta\, 2^{2k})(U^2 + \theta\, 2^{2k}U + (-4 + \theta^2)\, 2^{4k}).$$

The following facts of the field $\mathbb{K}$ are well known: the field discriminant is $148 = 2^2\, 37$, a $\mathbb{Z}$-basis for the ring integers $\mathcal{O}_\mathbb{K}$ is $\{1, \theta, \theta^2\}$, the class group is trivial, and the free part of the unit group of $\mathcal{O}_\mathbb{K}$ is generated by

$$\varepsilon_1 = -1 + \theta, \qquad \varepsilon_2 = 1 - 2\theta - \theta^2.$$

Further we have the following factorizations into prime ideals:

$$(2) = (\theta)^3, \qquad (-4 + 3\theta^2) = (\theta)^2\, (1 + \theta + \theta^2),$$

and we have

$$N\varepsilon_1 = N\varepsilon_2 = 1, \qquad N\theta = -2, \qquad N(1 + \theta + \theta^2) = 37.$$

Note that Mordell [M1] uses $-\varepsilon_1^2\varepsilon_2 = 2\theta - 1$ instead of ε_2 as second fundamental unit, and $4\theta - 3 = \varepsilon_1(1 + \theta + \theta^2)$ instead of $1 + \theta + \theta^2$.

Let δ be the squarefree part of $U - \theta\, 2^{2k}$. If a prime element $\pi \in \mathcal{O}_\mathbb{K}$ divides δ, it divides 2 or V^2. In the latter case even π^2 divides $2V^2$, and because δ is squarefree, π must divide the other factor of the right hand side of (1), $U^2 + \theta\, 2^{2k}U + (-4 + \theta^2)\, 2^{4k}$, too. But then π will divide any linear combination of $U - \theta\, 2^{2k}$ and $U^2 + \theta\, 2^{2k}U + (-4 + \theta^2)\, 2^{4k}$, in particular it will divide

$$(U^2 + \theta\, 2^{2k}U + (-4 + \theta^2)\, 2^{4k}) - (U + \theta\, 2^{2k+1})(U - \theta\, 2^{2k}) = (-4 + 3\theta^2)\, 2^{4k}.$$

In view of the above prime factorizations this leaves for π only the possibilities θ and $1 + \theta + \theta^2$, up to units.

Hence we can write

$$U - \theta\, 2^{2k} = \delta \times \text{a square.} \tag{2}$$

where

$$\delta = (-1)^a\, \varepsilon_1^b \varepsilon_2^c \theta^d (1 + \theta + \theta^2)^e$$

for some $a, b, c, d, e \in \{0, 1\}$, and the square is an algebraic integer, i.e. an element of $\mathcal{O}_{\mathbb{K}}$. For the norm of δ we have on the one hand

$$N\delta = (-1)^{a+d} 2^d 37^e,$$

and on the other hand, by (1), it differs by a rational integral square factor from

$$N(U - \theta 2^{2k}) = U^3 - 2^{4k+2}U + 2^{6k+1} = 2V^2.$$

It follows that $a = d = 1$ and $e = 0$. This leaves us four possibilities for δ, namely

$$\delta \in \{-\varepsilon_1\theta, -\varepsilon_1\varepsilon_2\theta, -\theta, -\varepsilon_2\theta\}.$$

At this point, to show that two of these four cases do not admit solutions, we use an argument that we find somewhat more elegant and more general than Mordell's arguments (on p. 1351 of [M1]). We study the three embeddings σ_1, σ_2, σ_3 of $\mathbb{K}$ into $\mathbb{R}$. They send θ to $\sigma_1(\theta) = -2.21...$, $\sigma_2(\theta) = 0.53...$, and $\sigma_3(\theta) = 1.67...$. Because

$$U^3 - 2^{4k+2}U + 2^{6k+1} = (U - 2^{2k}\sigma_1(\theta))(U - 2^{2k}\sigma_2(\theta))(U - 2^{2k}\sigma_3(\theta)) = 2V^2$$

has to be positive, we have two possibilities: either $U > 2^{2k}\sigma_3(\theta)$, or $2^{2k}\sigma_1(\theta) < U < 2^{2k}\sigma_2(\theta)$. Because by (2) for each $i \in \{1, 2, 3\}$ the sign of $U - 2^{2k}\sigma_i(\theta)$ has to be equal to the sign of $\sigma_i(\delta)$, we study the signs of these explicitly known numbers:

	θ	ε_1	ε_2	$-\varepsilon_1\theta$	$-\varepsilon_1\varepsilon_2\theta$	$-\theta$	$-\varepsilon_2\theta$
σ_1	$-$	$-$	$+$	$-$	$-$	$+$	$+$
σ_2	$+$	$-$	$-$	$+$	$-$	$-$	$+$
σ_3	$+$	$+$	$-$	$-$	$+$	$-$	$+$

This shows that in the case $U > 2^{2k}\sigma_3(\theta)$ it must be true that $\delta = -\varepsilon_2\theta$ (we call this *the first case*), and in the case $2^{2k}\sigma_1(\theta) < U < 2^{2k}\sigma_2(\theta)$ we must have $\delta = -\theta$ (we call this *the second case*).

The First Case

Let us first treat the case $U > 2^{2k}\sigma_3(\theta)$, thus $\delta = -\varepsilon_2\theta = -2 + 3\theta + 2\theta^2$. Making explicit the square in (2), for some $A, B, C \in \mathbb{Z}$ we have

$$U - \theta 2^{2k} = (-2 + 3\theta + 2\theta^2)(A + B\theta + C\theta^2)^2.$$

Working out the brackets and comparing coefficients, we find the following system of three quadratic equations:

$$A^2 + 3B^2 + 9C^2 + 3AB + 6AC + 8BC = 0, \tag{3}$$

$$3A^2 + 8B^2 + 20C^2 + 12AB + 16AC + 36BC = -2^{2k}, \tag{4}$$

$$-2A^2 - 6B^2 - 16C^2 - 8AB - 12AC - 24BC = U.$$

We may assume without loss of generality that A, B, C are coprime, and that $B \geqslant 0$.

From now on our proof diverges from Mordell's proof. We feel that for the situation we're in, with k not necessarily zero, our line of argument works prettier, but this is to some extent a matter of taste.

We view equation (3) as a quadratic equation in the variable A. Its discriminant should be a square, if rational solutions are to exist. Hence for a $D \in \mathbb{Z}$ we have

$$D^2 = (3B + 6C)^2 - 4(3B^2 + 8BC + 9C^2) = B(4C - 3B).$$

Here we are lucky, because the quadratic form in B, C in the right hand side factors over $\mathbb{Z}$. We let β be a prime divisor of the squarefree part of B. Then β divides also the squarefree part of $4C - 3B$, and since β divides both B and $4C - 3B$, we find that β divides $4C$. If β divides both B and C then (3) implies that β also divides A, and, in view of (4), we have $\beta = 2$. Using $B \geqslant 0$ our conclusion is that B is a square or twice a square.

In the case $B = E^2$ for an $E \in \mathbb{Z}$ (that we can assume to be nonnegative), also $4C - 3B$ is a square, say F^2, and we have $D = \pm EF$. We now solve (3) for A:

$$A = -\tfrac{3}{2}B - 3C \pm \tfrac{1}{2}D,$$

and express everything in E, F. In this way we find

$$A = \tfrac{1}{4}(-15E^2 \pm 2EF - 3F^2), \qquad B = E^2, \qquad C = \tfrac{1}{4}(3E^2 + F^2).$$

Since F is defined up to sign we may take the $\pm$-sign to be a $+$-sign. We insert the above expressions for A, B, C into equation (4), and obtain

$$\tfrac{1}{16}(-25E^4 + 12E^3F + 18E^2F^2 - 4EF^3 - F^4) = -2^{2k}.$$

We are lucky again, since the binary form in the left hand side of this equation factors over $\mathbb{Z}$, and we thus find

$$(E - F)(25E^3 + 13E^2F - 5EF^2 - F^3) = 2^m, \tag{5}$$

where $m = 2k + 4$. Had we not been this lucky, we would have arrived at a so-called Thue–Mahler equation. Procedures for solving such equations are known (see [TW2]), but are far from elementary.

Before studying this equation, we first mention that the second case, when $B = 2E^2$, leads to the same expressions for A, B, C in terms of E, F as above, multiplied by a factor 2. Hence we find the same quartic Eq. (5), but this time with $m = 2k + 2$.

Returning to Eq. (5), let us write $E = (-1)^g 2^h P$ and $F = (-1)^g 2^h Q$ for some nonnegative integers g, h, such that P, Q are coprime integers with $P > Q$. Now we proceed to solve

$$(P - Q)(25P^3 + 13P^2Q - 5PQ^2 - Q^3) = 2^n,$$

with $n = m - 4h$. Because $P - Q$ divides 2^n, there is an integer $l \geqslant 0$ such that $P - Q = 2^l$. Substituting $P = Q + 2^l$ into the above equation we find

$$32Q^3 + 2^{l+5}3Q^2 + 2^{2l+3}11Q + 2^{3l}25 = 2^{n-l}. \tag{6}$$

If $l = 0$ then (6) immediately yields that $n = 0$. So we have to solve

$$4Q^3 + 12Q^2 + 11Q + 3 = 0,$$

which is easily seen to have only $Q = -1$ as integral solution. It leads to $P = 0$, using the fact that A, B, C are coprime, further to $(E, F) = (0, \pm 2)$ with $m = 4$, and to $(A, B, C) = (-3, 0, 1)$ with $k = 0$. Finally, this gives $(U, V) = (2, \pm 1)$, and $(X, Y) = (1, 0), (1, -1)$.

If $l = 1$ then (6) becomes

$$32Q^3 + 192Q^2 + 352Q + 200 = 2^{n-1}.$$

The first terms $32Q^3$, $192Q^2$ and $352Q$ are all divisible by 32, whereas the last term 200 is only divisible by 8, and not anymore by 16. Hence the entire left hand side is divisible by 8 but not by 16, so $n - 1 = 3$, and we find the equation

$$Q^3 + 6Q^2 + 11Q + 6 = 0.$$

It has the solutions $Q = -3, -2, -1$ leading to $P = -1, 0, 1$. The case $(P, Q) = (-1, -3)$ leads to $(E, F) = (1, 3)$ with $m = 4$, and to $(A, B, C) = (-9, 1, 3)$ with $k = 0$. Finally, this gives $(U, V) = (12, \pm 29)$, and $(X, Y) = (6, 14), (6, -15)$. The case $(P, Q) = (0, -2)$ does not satisfy the requirements of P, Q being coprime (and is seen to lead to the solutions found above at $l = 0$). The case $(P, Q) = (1, -1)$ leads to $(E, F) = (1, -1)$ with $m = 4$, and to $(A, B, C) = (-5, 1, 1)$ with $k = 0$. Finally, this gives $(U, V) = (4, \pm 5)$, and $(X, Y) = (2, 2), (2, -3)$.

It remains to treat the case $l \geqslant 2$. This time in (6) the last three terms $2^{l+5}3Q^2$, $2^{2l+3}11Q$ and $2^{3l}25$ are divisible by 64, whereas the first term $32Q^3$ is only divisible by 32, but not by 64. It follows that $n - l = 5$. Note

that in Mordell's original work only $k = 0$ is treated, in which case $n \leqslant 4$, so that then the case $l \geqslant 2$ is trivial.

Putting, for convenience,

$$Z = Q + 2^l, \qquad W = 2^{l-2},$$

we find the equation

$$Z^3 - 4ZW^2 + 2W^3 = 1. \tag{7}$$

Equation (7) is a so-called Thue equation, that we conjecture to have only the following solutions:

$$(Z, W) = (1, 2), (1, 0), (-1, -1), (-5, -3), (-31, 14).$$

This can probably be proved by the deep methods of the Gelfond–Baker method, cf. [TW1]. But for us it would be like firing a cannon to kill a mosquito, because all we need is those solutions of (7) for which W is a power of 2. This can be done in an elementary way as follows.

First we show that if $|W| \geqslant 2$ then $|Z/W| < 2.61$. Namely, let θ_1, θ_2, θ_3 be the three roots of $t^3 - 4t + 2 = 0$ (thus the θ_i are the $\sigma_j(\theta)$ defined above, but not necessarily in the same order). The equation (7) now factors as

$$(Z - \theta_1 W)(Z - \theta_2 W)(Z - \theta_3 W) = 1,$$

and for a given solution Z, W we take indices such that

$$|Z - \theta_1 W| < |Z - \theta_2 W| < |Z - \theta_3 W|.$$

Either $|Z - \theta_1 W| \geqslant \frac{1}{2}|W| \min|\theta_i - \theta_j| > 0.567 |W|$, and then

$$1 = \prod_{i=1}^{3} |Z - \theta_i W| > |Z - \theta_1 W|^3 \geqslant (0.567 |W|)^3,$$

and then it follows that $|W| \leqslant 1$, or $|Z - \theta_1 W| < \frac{1}{2}|W| \min|\theta_i - \theta_j|$, and then for $k = 2, 3$ we find

$$|Z - \theta_k W| \geqslant |W| |\theta_1 - \theta_k| - |Z - \theta_1 W| > \tfrac{1}{2}|W| \min|\theta_i - \theta_j| > 0.567 |W|,$$

and thus by $|W| \geqslant 2$

$$\left| \frac{Z}{W} \right| \leqslant |\theta_1| + \left| \frac{Z}{W} - \theta_1 \right|$$

$$= |\theta_1| + \frac{1}{|Z - W\theta_2| \, |Z - W\theta_3| \, |W|} < 2.22 + \frac{1}{0.567^2 \, |W|^3} < 2.61.$$

Next we show that when $Z \neq 1$ and l is large, then so is $|Z/W|$. Namely, we look at (7) (mod 2^{2l-2}):

$$Z^3 = 4ZW^2 - 2W^3 + 1 = Z\,2^{2l-2} - 2^{3l-5} + 1 \equiv 1 \pmod{2^{2l-2}},$$

provided that $l \geqslant 3$, and it follows that

$$2^{2l-2} \,|\, (Z^3 - 1) = (Z-1)(Z^2 + Z + 1).$$

Since $Z^2 + Z + 1$ is always odd, we have $Z \equiv 1 \pmod{2^{2l-2}}$, hence $Z = 1$ or $|Z| \geqslant 2^{2l-2} - 1$. In the latter case we must have

$$\left|\frac{Z}{W}\right| \geqslant \frac{2^{2l-2} - 1}{2^{l-2}} = 2^l - \frac{1}{2^{l-2}}.$$

Putting things together, on noting that $2^l - 1/2^{l-2} < 2.61$ implies $l = 1$, we find for the case $l \geqslant 2$ only the possibilities $|W| \leqslant 1$ or $Z = 1$ (note that $l = 2$ implies $W = 1$). The solutions of (7) satisfying these conditions are easy to determine: the only one is $(Z, W) = (1, 2)$, with $l = 3$. It leads to $(P, Q) = (1, -7)$, and to $(E, F) = (1, -7)$ with $m = 8$, further to $(A, B, C) = (-44, 1, 13)$ with $k = 2$, to $(U, V) = (322, \pm 4066)$, and finally to $(X, Y) = (161/16, -2065/64)$, $(161/16, 2001/64)$.

The Second Case

Now we treat the case $2^{2k}\sigma_1(\theta) < U < 2^{2k}\sigma_2(\theta)$ where $\delta = -\theta$. Note that in Mordell's original work only $k = 0$ is treated, in which case we have at once $-2 \leqslant U \leqslant 1$, which is trivial.

We proceed as in the first case above. So for some $A, B, C \in \mathbb{Z}$ we have

$$U - \theta\,2^{2k} = -\theta(A + B\theta + C\theta^2)^2.$$

Working out the brackets and comparing coefficients, we find the following system of three quadratic equations:

$$-C^2 + AB + 4BC = 0, \tag{8}$$

$$-A^2 - 4B^2 - 16C^2 - 8AC + 4BC = -2^{2k}, \tag{9}$$

$$2B^2 + 8C^2 + 4AC = U.$$

We may assume without loss of generality that A, B, C are coprime, and that $B \geqslant 0$.

We are lucky once more, in that equation (8) now gives at once

$$(C - 2B)^2 = B(A + 4B),$$

so that again B is a square or twice a square.

In the case $B = E^2$ (with an E that we can assume to be nonnegative) we have $A + 4B = F^2$, and we may take $C = 2E^2 + EF$. We substitute this into (9), and thus obtain

$$12E^4 + 28E^3F + 24E^2F^2 + 8EF^3 + F^4 = 2^m \tag{10}$$

with $m = 2k$. And again, in the case $B = 2E^2$ we find the same equation (10), but with $m = 2k - 2$.

This time the binary form in the left hand side of (10) does not factor over $\mathbb{Z}$, so now we seem to have run out of luck, and have to turn to non-elementary methods such as [TW2]. But fortunately this is not so. To start with, if $m \geqslant 1$ then F is even, say $F = 2F_1$. Hence

$$2^{m-2} = 3E^4 + 14E^3F_1 + 24E^2F_1^2 + 16EF_1^3 + 4F_1^4,$$

and we see that if $m \geqslant 3$ then also E is even. Since A, B and C are coprime integers, we have only to search the solutions with $m \leqslant 2$.

Further, our luck is that (10) does not have any linear factors over $\mathbb{R}$. Using this, we observe that $x^4 + 8x^3 + 24x^2 + 28x + 12$ has as minimal value 1 (at $x = -1$), and then by (10) we get

$$2^m = E^4\left(\left(\frac{F}{E}\right)^4 + 8\left(\frac{F}{E}\right)^3 + 24\left(\frac{F}{E}\right)^2 + 28\frac{F}{E} + 12\right) \geqslant E^4.$$

But then we see $E^4 \leqslant 2^m \leqslant 4$, hence $|E| \leqslant 1$. Now it is easily seen that in fact there are only three solutions: $(E, F) = (0, \pm 1), (1, -1), (1, -2)$. The case $(E, F) = (0, \pm 1)$ with $m = 0$ leads to $(A, B, C) = (1, 0, 0)$ with $k = 0$, further to $(U, V) = (0, \pm 1)$, and finally to $(X, Y) = (0, 0), (0, -1)$. The case $(E, F) = (1, -1)$ with $m = 0$ leads to $(A, B, C) = (-3, 1, 1)$ with $k = 0$, further to $(U, V) = (-2, \pm 1)$, and finally to $(X, Y) = (-1, 0), (-1, -1)$. The case $(E, F) = (1, -2)$ with $m = 2$ leads to $(A, B, C) = (0, 1, 0)$ with $k = 1$, further to $(U, V) = (2, \pm 2)$, and finally to $(X, Y) = (\frac{1}{4}, \frac{-5}{8}), (\frac{1}{4}, \frac{-3}{8})$.

This completes the proof. ∎

REFERENCES

[AEH] H. L. Abbott, P. Erdös, and D. Hanson, On the number of times an integer occurs as a binomial coefficient, *Am. Math. Monthly* **81** (1974), 256–261.

[Av]   È. T. Avanesov, Solution of a problem on figurative numbers [in Russian], *Acta Arith.* **12** (1966/67), 409–420.

[F]   G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlköpern, *Invent. Math.* **73** (1983), 349–366.

[G]   R. K. Guy, "Unsolved Problems in Number Theory," Second edition, Springer-Verlag, New York, 1994.

[GPZ] J. Gebel, A. Pethö, and H. G. Zimmer, Computing integral points on elliptic curves, *Acta Arith.* **68** (1994), 171–192.

[L]   D. A. Lind, The quadratic field $\mathbb{Q}(\sqrt{5})$ and a certain diophantine equation, *Fibonacci Quart.* **6** (1968), 86–93.

[M1]   L. J. Mordell, On the integer solutions of $y(y+1) = x(x+1)(x+2)$, *Pacific J. Math.* **13** (1963), 1347–1351.

[M2]   L. J. Mordell, "Diophantine Equations," Academic Press, New York, 1969.

[Pi]   Ákos Pintér, A note on the diophantine equation $(\begin{smallmatrix}x\\4\end{smallmatrix}) = (\begin{smallmatrix}y\\2\end{smallmatrix})$, *Publ. Math. Debrecen* **47** (1995), 411–415.

[Sil]   J. H. Silverman, "The arithmetic of Elliptic Curves," Springer-Verlag, Berlin, 1986.

[Sin1] D. Singmaster, How often does an integer occur as a binomial coefficient, *Am. Math. Monthly* **78** (1971), 385–386.

[Sin2] D. Singmaster, Repeated binomial coefficients and Fibonacci numbers, *Fibonacci Quart.* **13** (1975), 295–298.

[Sm]   N. P. Smart, S-integral points on elliptic curves, *Math. Proc. Cambridge Phil. Soc.* **116** (1994), 391–399.

[ST]   R. J. Stroeker and N. Tzanakis, Solving elliptic diophantine equations by estimating linear forms in elliptic logarithms, *Acta Arith.* **67** (1994), 177–196.

[TW1]   N. Tzanakis and B. M. M. de Weger, On the practical solution of the Thue equation, *J. Number Theory* **31** (1989), 99–132.

[TW2]   N. Tzanakis and B. M. M. de Weger, How to explicitly solve a Thue–Mahler equation, *Compositio Math.* **84** (1992), 223–288.

[dW]   B. M. M. de Weger, A binomial diophantine equation, *Quart. J. Math. Oxford Ser. 2* **47** (1996), 221–231.